

On “No-go theorem for slowly rotating black holes in Hořava-Lifshitz gravity”

Anzhong Wang*

GCAP-CASPER, Physics Department, Baylor University, Waco, TX 76798-7316, USA

(Dated: December 6, 2012)

Slowly rotating black holes in the non-projectable Hořava-Lifshitz (HL) theory were studied recently in Phys. Rev. Lett. **109**, 181101 (2012), and claimed that they do not exist. In this Comment, we show that this is incorrect, and such solutions indeed exist in the IR limit of the non-projectable HL theory.

I. INTRODUCTION

Recently, Barausse and Sotiriou [1] studied slowly rotating black holes in the non-projectable version [2] of the Horava-Lifshitz (HL) theory [3], by using the equivalence [4] (See also [2]) between the IR limit of the non-projectable HL theory and the Einstein-aether (æ-) theory with the hypersurface-orthogonal condition, and claimed that such black holes do not exist. This is a very strong statement, and immediately causes cautions on the variability of this version of the HL theory, because observations indicate that such black holes very likely exist in our universe [5].

In this Comment, we show that the above claim is incorrect, as it was based on three wrong field equations. After correcting these errors, we find that slowly rotating black holes indeed exist in the infrared (IR) limit of the non-projectable HL theory [2].

To this purpose, let us start with the hypersurface-orthogonal æ-theory (For detail, see [4]). The fundamental variables of the gravitational sector in the æ-theory are $(g_{\mu\nu}, u^\lambda)$, where Greek letters run from 0 to 3, $g_{\mu\nu}$ is the four-dimensional metric of the space-time with the signatures $(+, -, -, -)$, and u^λ the aether four-velocity. The general action of the theory is given by, $S = S_\text{æ} + S_m$, where S_m denotes the action of matter, and $S_\text{æ}$ the gravitational action of the æ-theory, given by

$$\begin{aligned} S_\text{æ} &= \zeta^2 \int \sqrt{-g} d^4x \left[-R(g_{\mu\nu}) + \mathcal{L}_\text{æ}(g_{\mu\nu}, u^\lambda) \right], \\ S_m &= \int \sqrt{-g} d^4x \left[\mathcal{L}_m(g_{\mu\nu}, \psi) \right]. \end{aligned} \quad (1)$$

Here $\zeta^2 = 1/(16\pi G_\text{æ})$, ψ collectively denotes the matter fields, R and g are, respectively, the Ricci scalar and determinant of $g_{\mu\nu}$, and

$$\mathcal{L}_\text{æ} = -M^{\alpha\beta}{}_{\mu\nu} (D_\alpha u^\mu) (D_\beta u^\nu), \quad (2)$$

where D_μ denotes the covariant derivative with respect to $g_{\mu\nu}$, and $M^{\alpha\beta}{}_{\mu\nu}$ is defined as

$$M^{\alpha\beta}{}_{\mu\nu} = c_1 g^{\alpha\beta} g_{\mu\nu} + c_2 \delta_\mu^\alpha \delta_\nu^\beta + c_3 \delta_\nu^\alpha \delta_\mu^\beta + c_4 u^\alpha u^\beta g_{\mu\nu}. \quad (3)$$

Note that here we assume that matter fields couple only to $g_{\mu\nu}$, so \mathcal{L}_m is independent of u^μ . The four coupling constants c_i are all dimensionless, and $G_\text{æ}$ is related to the Newtonian constant G_N via the relation, $G_N = 2G_\text{æ}/(2 - c_{14})$, with $c_{14} \equiv c_1 + c_4$, etc. The hypersurface-orthogonal condition,

$$\omega_\mu \equiv \epsilon_{\mu\nu\alpha\beta} u^\nu D^\alpha u^\beta = 0, \quad (4)$$

implies that there exists a time-like scalar function T , so that u_μ is given by [6],

$$u_\mu = T_{,\mu} / |g^{\alpha\beta} T_{,\alpha} T_{,\beta}|^{1/2}, \quad (5)$$

where the leaves of constant T naturally provides the foliations constructed in the HL theory [3]. Then, it is very convenient to choose T as the time-like coordinate, and with the ADM decompositions [7], the aether four-velocity can be expressed as,

$$u_\mu = N \delta_\mu^T, \quad u^\mu = \frac{1}{N} (\delta_T^\mu - N^i \delta_i^\mu), \quad (i, j = 1, 2, 3) \quad (6)$$

where $N_i = h_{ij} N^j$, $h_{ij} h^{ik} = \delta_j^k$, and N , N^i and h_{ij} are respectively, the lapse function, shift vector and 3-metric defined on the leaves. Then, we find that

$$\delta S = \zeta^2 \int \left[(E^{\mu\nu} - 8\pi G_\text{æ} T^{\mu\nu}) \delta g_{\mu\nu} + \mathcal{E}_\mu \delta u^\mu \right], \quad (7)$$

where $T^{\mu\nu} \equiv -(2\delta(\sqrt{-g}\mathcal{L}_m)/\delta g_{\mu\nu})/\sqrt{-g}$, and

$$\begin{aligned} E^{\mu\nu} &= R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - T_\text{æ}^{\mu\nu}, \\ T_\text{æ}^{\alpha\beta} &= D_\mu \left[u^{(\beta} J^{\alpha)\mu} - J^{\mu(\alpha} u^{\beta)} - J^{(\alpha\beta)} u^\mu \right] \\ &\quad + c_1 \left[(D_\mu u^\alpha) (D^\mu u^\beta) - (D^\alpha u_\mu) (D^\beta u^\mu) \right] \\ &\quad + c_4 a^\alpha a^\beta - \frac{1}{2} \mathcal{L}_\text{æ} g^{\alpha\beta}, \\ \mathcal{E}_\mu &= \frac{\delta \mathcal{L}_\text{æ}}{\delta u^\mu} = 2 \left(D_\alpha J^\alpha{}_\mu - c_4 a_\alpha D_\mu u^\alpha \right), \end{aligned} \quad (8)$$

with $J^\alpha{}_\mu \equiv M^{\alpha\beta}{}_{\mu\nu} D_\beta u^\nu$ and $a^\mu \equiv u^\alpha D_\alpha u^\mu$. It should be noted that $T_\text{æ}^{\alpha\beta}$ defined above is different from the one given in [8] by a term $\lambda_\text{æ} u^\alpha u^\beta$. The definition of \mathcal{E}_μ is also different.

In the æ-theory, $g_{\mu\nu}$ and u^μ are independent, and the variations of S with respect to them yield, respectively,

*Electronic address: anzhong.wang@baylor.edu

the Einstein-aether equations, $E^{\mu\nu} = 8\pi G_{\text{ae}} T^{\mu\nu}$, and the aether equation $\mathcal{A}_\mu = 0$. However, the hypersurface-orthogonal condition relates u^μ with the metric components through Eq.(6). Then, we obtain,

$$\begin{aligned}\delta u^\mu &= \frac{N^i \delta_i^\mu - \delta_T^\mu}{N^2} \delta N - \frac{\delta_i^\mu}{N} \delta N^i, \quad \frac{\delta g_{\mu\nu}}{\delta N} = 2N \delta_\mu^T \delta_\nu^T, \\ \frac{\delta g_{\mu\nu}}{\delta N^i} &= -2N_i \delta_\mu^T \delta_\nu^T - 2h_{ij} \delta_\mu^{(T} \delta_\nu^{j)}, \\ \frac{\delta g_{\mu\nu}}{\delta h_{ij}} &= -N^i N^j \delta_\mu^T \delta_\nu^T - N^i \delta_\mu^{(T} \delta_\nu^{j)} - N^j \delta_\mu^{(T} \delta_\nu^{i)} - \delta_\mu^{(i} \delta_\nu^{j)},\end{aligned}\quad (9)$$

where $f_{(ij)} \equiv (f_{ij} + f_{ji})/2$. Substituting the above expressions into Eq.(7), we find that the variations of S with respect to N, N^i and h_{ij} yield, respectively, the Hamiltonian, momentum constraints, and the dynamical equations, given by,

$$\mathcal{H}^\perp = 8\pi G_{\text{ae}} \rho_H, \quad (10)$$

$$\mathcal{H}_i = 8\pi G_{\text{ae}} s_i, \quad (11)$$

$$\mathcal{H}^{ij} = 8\pi G_{\text{ae}} s^{ij}, \quad (12)$$

where

$$\begin{aligned}\mathcal{H}^\perp &\equiv E^{\mu\nu} \frac{\delta g_{\mu\nu}}{\delta N} + \mathcal{A}_\mu \frac{\delta u^\mu}{\delta N}, \quad \rho_H \equiv T^{\mu\nu} \frac{\delta g_{\mu\nu}}{\delta N}, \\ \mathcal{H}_i &\equiv E^{\mu\nu} \frac{\delta g_{\mu\nu}}{\delta N^i} + \mathcal{A}_\mu \frac{\delta u^\mu}{\delta N^i}, \quad s_i \equiv T^{\mu\nu} \frac{\delta g_{\mu\nu}}{\delta N^i}, \\ \mathcal{H}^{ij} &\equiv E^{\mu\nu} \frac{\delta g_{\mu\nu}}{\delta h_{ij}}, \quad s_{ij} \equiv T^{\mu\nu} \frac{\delta g_{\mu\nu}}{\delta h_{ij}}.\end{aligned}\quad (13)$$

When $E^{\mu\nu} = \mathcal{A}_\mu = 0$, we find that

$$\mathcal{H}^\perp = \mathcal{H}_i = \mathcal{H}^{ij} = 0, \quad (E^{\mu\nu} = \mathcal{A}_\mu = 0). \quad (14)$$

That is, if $(g_{\mu\nu}, u^\mu)$ is a vacuum solution of the hypersurface-orthogonal æ-theory, it is also a vacuum solution (N, N^i, h_{ij}) of the IR limit of the non-projectable HL theory, as shown explicitly in [4], although conversely it does not hold in general.

It is also interesting to note that one of the four coupling constants c_i can be eliminated by the field redefinitions [9],

$$g'_{\mu\nu} = g_{\mu\nu} + (\sigma - 1)u_\mu u_\nu, \quad u'^\mu = \frac{1}{\sqrt{\sigma}} u^\mu, \quad (15)$$

for which the action (1) remains the same in terms of $g'_{\mu\nu}$, after the replacements, $c_i \rightarrow c'_i$ [9], where σ is a positive constant. On the other hand, it can be shown that

$$\omega^\mu \omega_\mu = a^2 - (D_\alpha u_\beta)(D^\alpha u^\beta) + (D_\alpha u_\beta)(D^\beta u^\alpha), \quad (16)$$

where $a^2 \equiv a^\mu a_\mu$. Then, in the hypersurface-orthogonal case, one can always add a term,

$$\Delta S_{\text{ae}} = \frac{1}{16\pi G_{\text{ae}}} \int \sqrt{-^{(4)}g} d^4x (\alpha \omega^\mu \omega_\mu), \quad (17)$$

into S , which is effectively to shift the coupling constants c_i to c'_i , where $c'_1 = c_1 + \alpha$, $c'_2 = c_2$, $c'_3 = c_3 - \alpha$, $c'_4 = c_4 - \alpha$. With the above gauge freedom, one can see that in the hypersurface-orthogonal case, there are essentially only two independent coupling constants.

II. SLOWLY ROTATING SPACETIMES

Setting a spherical body into slow and uniform rotation about an axis, one expects that the metric outside the body will change from its spherically symmetric geometry to a stationary and axisymmetric configuration. In this regard, let us consider the spacetimes, described by the metric,

$$\begin{aligned}ds^2 &= N^2(r)dt^2 - \frac{1}{f(r)} \left(dr + h(r)dt \right)^2 - r^2 d\Omega^2 \\ &\quad - 2r^2 \sin^2 \theta \omega(r, \theta) dt d\phi,\end{aligned}\quad (18)$$

where we assume that $(N(r), f(r), h(r))$ denotes the static spherical vacuum solutions of the æ-theory without rotation, and “slowly rotating” means that

$$|\omega| \ll 1. \quad (19)$$

It can be shown that for the metric (18) $\zeta_{(t)} = \partial_t$ is still a timelike Killing vector, while $\zeta_{(\phi)} = \partial_\phi$ a space-like Killing vector with closed orbit. So, it indeed represents stationary axisymmetric spacetimes.

In addition, the 4-velocity defined by Eq.(6) is still hypersurface-orthogonal even for the metric (18). One might expect that the aether may also rotate with respect to the chosen frame, so that it has a non-vanishing ϕ component, i.e.,

$$u_\mu = N \delta_\mu^t + u_3(r, \theta) \delta_\mu^\phi. \quad (20)$$

While in general this is indeed true, the hypersurface-orthogonal condition (4) requires

$$\frac{u'_3}{u_3} - \frac{N'}{N} = 0, \quad u_{3,\phi} = 0, \quad (21)$$

for which we have $u_3 = \ell N$, where ℓ is a constant. However, this equivalent to the coordinate transformation, $t \rightarrow t + \ell\phi$, and the slowly rotating condition requires $\ell \simeq \mathcal{O}(\omega) \ll 1$. Then, this coordinate transformation will leave the metric (18) unchanged, after a redefinition of ω [1]. Thus, without loss of generality, we consider only the case where $u_3 = 0$.

To process further, we note that the physics is independent of the gauge, given by Eqs.(15) and (17). Thus, without loss of generality, in the following we set

$$\hat{c}_3 = -\hat{c}_1, \quad \hat{c}_4 = 0, \quad (22)$$

where quantities with hats denote the ones after performing consequentially the two operations given by Eqs.(15) and (17). This is equivalent to the choice

$$\sigma = \frac{1}{1 - c_+}, \quad \alpha = \frac{2c_4 + c_+ c_-}{2(1 - c_+)}, \quad (23)$$

if one first considers the operation (17) and then the one (15). Our choice (22) is the same as that given by Eq.(15) in [4]. In terms of c_i , we have

$$\hat{c}_1 = \frac{1}{2} (2c_4 + c_+ + 1), \quad \hat{c}_2 = \frac{c_{123}}{1 - c_+}. \quad (24)$$

It is interesting to note that c_{14} is invariant under both of the operations, and is irrelevant with their ordering,

$$c_{14} = c'_{14} = c''_{14} = \hat{c}_{14}. \quad (25)$$

Then, to the zeroth-order of ω , we obtain the spherical hypersurface-orthogonal Einstein-aether field equations from Eqs.(10), (11) and (12),

$$\bar{\mathcal{H}}_3 = 8\pi G_{\text{æ}} \bar{\rho}_H, \quad (26)$$

$$\bar{\mathcal{H}}_i = 8\pi G_{\text{æ}} \bar{s}_i, \quad (27)$$

$$\bar{\mathcal{H}}^{ij} = 8\pi G_{\text{æ}} \bar{s}^{ij}, \quad (28)$$

where quantities with bars denote the spherical seed solutions.

To the first-order of ω , we find that

$$\delta\mathcal{H}^\perp = 0, \quad (29)$$

while the non-vanishing components of $\delta\mathcal{H}_i$ and $\delta\mathcal{H}^{ij}$ are given by

$$\begin{aligned} \delta\mathcal{H}_3 &= -\frac{\sin^2 \theta}{N^2} \left[\frac{(\sin^3 \theta \omega_{,\theta})_{,\theta}}{\sin^3 \theta} + \frac{N\sqrt{f}}{r^2} \left(\frac{r^4 \sqrt{f}}{N} \omega' \right)' \right], \\ \delta\mathcal{H}^{13} &= -\frac{\sqrt{f}h}{2r^4 N} \left(\frac{r^4 \sqrt{f}}{N} \omega' \right)', \\ \delta\mathcal{H}^{23} &= -\frac{\sqrt{f}}{2r^4 N} \left[\frac{r^2 h \omega_{,\theta}}{N\sqrt{f}} + \omega \left(\frac{r^2 h}{N\sqrt{f}} \right)' \right]_{,\theta}. \end{aligned} \quad (30)$$

In the vacuum case, we have

$$\delta\mathcal{H}^\perp = \delta\mathcal{H}_i = \delta\mathcal{H}^{ij} = 0. \quad (31)$$

When $h = 0$, from the above expressions we can see that $\delta\mathcal{H}^{ij} = 0$ is satisfied identically, and the momentum constraint $\delta\mathcal{H}_i = 0$ yields,

$$\frac{(\sin^3 \theta \omega_{,\theta})_{,\theta}}{\sin^3 \theta} + \frac{N\sqrt{f}}{r^2} \left(\frac{r^4 \sqrt{f}}{N} \omega' \right)' = 0, \quad (h = 0), \quad (32)$$

which can be easily solved by variable separation methods. Since it is not related to black holes, we shall not consider this case any further.

When $h \neq 0$, Eq.(31) yields

$$(\sin^3 \theta \omega_{,\theta})_{,\theta} = 0, \quad (33)$$

$$\left(\frac{r^4 \sqrt{f}}{N} \omega' \right)' = 0, \quad (34)$$

$$\left[\omega' + \left(\frac{h'}{h} - \frac{f'}{2f} - \frac{N'}{N} + \frac{2}{r} \right) \omega \right]_{,\theta} = 0. \quad (35)$$

From Eq.(33) we obtain

$$\omega(r, \theta) = \omega_2(r) \left[\frac{\cos \theta}{\sin^2 \theta} - \ln \left(\tan \frac{\theta}{2} \right) \right] + \omega_1(r), \quad (36)$$

which is singular at $\theta = 0, \pi$, unless $\omega_2(r) = 0$. Then, substituting it into Eq.(34), we obtain

$$\omega = -3J \int \frac{N dr}{r^4 \sqrt{f}}, \quad (37)$$

for which Eq.(35) is satisfied identically. Asymptotical flatness condition requires $f \simeq N^2 \sim 1$, and from the above we find that

$$\omega \simeq \frac{J}{r^3}, \quad (r \gg 1). \quad (38)$$

Note that the slowly rotating Kerr black hole has the same limit.

Therefore, it is concluded that, *for any given spherical vacuum solution $(N(r), f(r), h(r))$ of the æ-theory with the hypersurface-orthogonal condition, there always exists a solution $(N(r), f(r), h(r), \omega(r, \theta))$, which represents a slowly rotating vacuum solution of the T-theory. When the rotation is switched off, it reduces to the seed solution $(N(r), f(r), h(r))$, where ω is given by Eq.(??) for $h = 0$, and by Eq.(37) for $h \neq 0$.*

Note that Eqs.(33)-(35) are quite different from Eqs.(11) - (13) given in [1]. Hence, contrary conclusions regarding to the existence of slowly rotating black holes were obtained.

III. SPHERICAL STATIC BLACK HOLES IN EINSTEIN-AETHER THEORY

Barausse, Jacobson and Sotiriou (BJS) recently studied spherical static vacuum spacetimes in the framework of the æ-theory, and found numerically a class of solutions that represents black holes. BJS chose to work with the Eddington-Finkelstein coordinates [8],

$$ds^2 = F(r)dv^2 - 2B(r)dvdr - r^2 d\Omega^2, \quad (39)$$

in which the four-velocity of the aether is given by

$$u^\alpha \partial_\alpha = A(r) \partial_v - \frac{1 - F(r)A^2(r)}{2A(r)B(r)} \partial_r. \quad (40)$$

In the spherical case, since the aether is always hypersurface-orthogonal [4], one can introduce the time-like variable T , so that u_α in terms of T takes the form (5), from which we find that

$$dv = \frac{dT}{T_{,v}} + \frac{2A^2 B}{1 + A^2 F} dr. \quad (41)$$

The integrability condition requires $T_{,vr} = 0$. Without loss of generality, we choose $T_{,v} = 1$, so that Eq.(41) has the solution,

$$v = T + \int^r \left(\frac{2A^2 B}{1 + A^2 F} \right) dr. \quad (42)$$

Inserting it into Eq.(39), we find that the metric becomes

$$ds^2 = N^2(r)dT^2 - \frac{1}{f(r)}\left(dr + h(r)dT\right)^2 - r^2 d\Omega^2, \quad (43)$$

with

$$\begin{aligned} N(r) &= \frac{1 + A^2 F}{2A}, \quad f(r) = \left(\frac{1 + A^2 F}{2AB}\right)^2, \\ h(r) &= \frac{1 - A^4 F^2}{4A^2 B}. \end{aligned} \quad (44)$$

Instead of choosing the gauge(22), BJS imposed the conditions [1, 8],

$$\sigma = s_0^2 = \frac{c_{123}(2 - c_{14})}{c_{14}(1 - c_+)(2 + c_+ + 3c_2)}, \quad \tilde{c}_4 = 0. \quad (45)$$

Here s_0 is the speed of the spin-0 mode, so that the spin-0 horizon coincides with the metric horizon of the redefined metric $g'_{\mu\nu}$.

With the above, BJS were able to show that there exists a class of black hole solutions, characterized by the radii of their horizons, which is asymptotically flat and free of space-time singularities (except at the origin $r = 0$). Using the field redefinitions, one can transform the BJS solutions to the gauge (22). Since $u_\mu = N\delta_\mu^T$, it is clear that these field redefinitions only change the lapse function from N to $\sqrt{\sigma}N$, while N^i and h_{ij} remain the same. As a result, even in the gauge (22), the BJS black hole solutions still take the form (43). Note that these black hole solutions were rederived in [10].

Taking these black hole solutions as the seeds, from the results presented in the previous section one can see that

slowly rotation black hole solutions indeed exist in the IR limit of the non-projectable HL theory [2], in contrast to the claim presented in [1].

IV. CONCLUSIONS

In this Comment, we have studied slowly rotating spacetimes, including black holes, in the non-projectable HL theory [2], by using the equivalence between this version of the HL theory in its IR limit and the æ-theory with hypersurface-orthogonal conditions [4]. We have found that slowly rotating black holes indeed exist. This is in contrast to the results obtained in [1].

It should be noted that the equivalence between the æ-theory and the IR limit of the non-projectable HL theory holds only in the level of action. In particular, the æ-theory still has the general diffeomorphisms as that of general relativity, while the HL theory has only the foliation-preserving diffeomorphisms. It is exactly because of the former that we are allowed to make coordinate transformations of the kind $T = T(v, r)$, which are forbidden by the foliation-preserving diffeomorphisms [3].

In addition, it was shown that the static spherical black holes are not stable against non-spherical perturbations [10], although it is not clear whether or not the high order derivative terms will fix this instability.

Acknowledgements

The author would like to thank E. Barausse, T. Jacobson, S. Sibiryakov and T. Sotiriou for valuable discussions and comments. This work is supported in part by the DOE Grant, DE-FG02-10ER41692.

-
- [1] E. Barausse and T.P. Sotiriou, Phys. Rev. Lett. **109**, 181101 (2012) [arXiv:1207.6370].
 - [2] D. Blas, O. Pujolas, and S. Sibiryakov, JHEP **04**, 018 (2011).
 - [3] P. Hořava, Phys. Rev. D **79**, 084008 (2009).
 - [4] T. Jacobson, Phys. Rev. D **81**, 101502 (2010) [arXiv:1001.4823].
 - [5] R. Narayan, New J. Phys. **7**, 199 (2005).
 - [6] R.M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
 - [7] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (W.H. Freeman and Company, San Francisco, 1973), pp.484-528.
 - [8] E. Barausse, T. Jacobson, and T.P. Sotiriou, Phys. Rev. D **83**, 124043 (2011).
 - [9] B.Z. Foster, Phys. Rev. D **72**, 044017 (2005).
 - [10] D. Blas and S. Sibiryakov, Phys. Rev. D **84**, 124043 (2011).